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Brief paper

Stability of finite horizon model predictive control with incremental input constraints☆

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ABSTRACT

Model predictive control of discrete-time nonlinear systems with incremental input constraints is proposed in this paper. Firstly, the existence of the terminal set and terminal penalty is proven on the assumption that the considered system is twice continuously differentiable. Secondly, properties of the optimal cost function are exploited. It shows that the optimal cost function is positive semi-definite, continuous at the equilibrium and monotonically decreasing along the predicted trajectory. The systems under control converge to the equilibrium since the optimal cost function is monotonically decreasing. Thirdly, stability of nonlinear systems is proven in terms of the classical Lyapunov Theorem, where an upper bound of the optimal cost function in the terminal set is chosen as a candidate Lyapuonv function. Finally, the system is asymptotically stable since the system state converges to the equilibrium and the system is stable.

1. Introduction

The control action of model predictive control (MPC) is computed online to predict system behaviors over a time horizon, where a certain cost function is minimized (Chen & Allgöwer, 1998; Mayne, Rawlings, Rao, & Scokaert, 2000). The obtained optimal input is then applied until the next sampling instant, and the procedure is repeated again with a new measurement. The main difference between MPC and classic control schemes is that a control sequence rather than a control law is determined at each time instant.

A constraint is anything that prevents the system from achieving its goal. There are three major types of constraints frequently encountered in applications, i.e., constraints on control incremental variation, constraints on control amplitude variation, and constraints on output or state (Wang, 2009). If constraints of the considered systems are not paid attention to, the closed loop control performance could be severely deteriorated (Chen, 1997) in the presence of constraints. Since it is easy to include all kinds of equality and inequality constraints in the optimization problem, MPC is one of the most effective schemes to deal with constraints.

The change rate of variations of real systems is bounded mostly due to the limits of physics or modeling assumptions. Beyond the limits, either the model describing real systems is invalid or the system performance is deteriorated. In particular, incremental input constraints are serious challenges in many automatic control applications, which can induce a considerable destabilizing effect due to phase-lag (Angeli, Casavola, & Mosca, 2000; Berg, Hammet, Schwartz, & Banda, 1996). Joint constraints on both input magnitude and increment are considered in Tyan and Bernstein (1997) for a system consisting of a chain of cascade integrators. It is shown that a linear system subject to both the actuator position and the rate saturation is semi-globally stabilizable by linear state feedback control law, if it is asymptotically null-controllable with bounded controls (Lin, 1997). Control of linear systems with control constraint rate or increment with additive bounded disturbances is considered in Mesquine, Tadeo, and Benzaouia (2004, 2006), where necessary and sufficient conditions that the system evolution respects rate or increment constraints are used to derive stabilizing feedback control. Problems of designing stabilizing regulators for linear systems subject to control saturations and asymmetric constrains on its increment or

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rate are solved in Benzaouia, Tadeo, and Mesquine (2006), which are valid for asymmetric constraints on the increment or rate of the control. Predictive control of linear systems in the presence of joint positional and incremental input saturation constraints is developed in Angeli et al. (2000), Mosca and Tesi (2008) and Mosca, Tesi, and Zhang (2008), where set-point tracking and constant disturbances rejection problem are considered. Infinite horizon MPC with incremental constraints is analyzed in Ding (2009), Gonzalez and Odloak (2011), Gonzalez, Perez, and Odloak (2009) and Rodrigues and Odloak (2003) for linear systems, where the infinite horizon controllers are reduced to finite horizon controllers by defining a terminal penalty. It is emphasized that all these contributions deal mainly with the linear systems. Infinite horizon MPC of nonlinear systems with incremental constraints is proposed in Yu, Qu, and Chen (2013), where convergence rather than asymptotic stability is discussed. MPC schemes of nonlinear systems with guaranteed stability consider state constraints, input constraints and output constraints in their formulation. However, in general, incremental input constraints are not taken into account in Chen and Allgöwer (1998), Mayne et al. (2000), Rawlings and Muske (1993) and Yu, Qu, Xu, and Chen (2015).

In this paper, finite horizon MPC of nonlinear systems with state constraints, input constraints and incremental input constrains is proposed. Firstly, the existence of the terminal set, terminal penalty and terminal control law is proven under the assumption that the considered nonlinear systems is twice continuously differentiable. An extra terminal inequality is imposed on the optimization problem in order to guarantee recursive feasibility. Secondly, the properties of the optimal cost function of the finite horizon MPC are exploited, which are continuous at the equilibrium, positive semi-definite and monotonically decreasing along the predicted trajectory. Furthermore, the terminal penalty which is an upper bound of the optimal cost function in the terminal set is chosen as a candidate Lyapunov function. Thus, stability of the finite horizon MPC is proven in accordance with the classical Lyapunov Theorem. Finally, asymptotic stability is claimed since the system state converges to the equilibrium, and the system is stable.

The organization of this paper is as follows. The problem setup is introduced in Section 2. The properties of the optimal cost function as well as recursive feasibility of the optimization problem are discussed in Section 3. Asymptotic stability of finite horizon MPC of nonlinear systems is discussed in Section 4. A simulation example is given in Section 5. Some concluding remarks are made in Section 6.

1.1. Notations and basic definitions

Let $\mathbb{R}$ denote the field of real numbers and $\mathbb{R}^n$ the n-dimensional Euclidean space, $\mathbb{Z}$ the field of non-negative integers, $\mathbb{Z}_+$ the field of positive integers, $k+i|k$ the predicted value at the time instant $k+i$ starting from the time instant k. For a vector $v \in \mathbb{R}^n$, $\|v\|$ denotes the 2-norm and $\|v\|_Q = \sqrt{v^T Q v}$ with $Q \in \mathbb{R}^{n\times n}$ and $Q > 0$. The matrix I denotes the identity matrix with compatible dimension. For $M \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(M)$ ($\lambda_{\max}(M)$) is the smallest (largest) real part of the eigenvalues of matrix M and $\sigma(M)$ the largest singular value of M.

2. Problem setup

Consider discrete-time nonlinear systems:

$$x_{k+1} = f(x_k, u_k), \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$ are the system state and control input at the time instant k, respectively. Denote the input increment as Δu_k, where $\Delta u_k = u_k - u_{k-1}$ for all $k \geq 1$, and $\Delta u_0 = u_0$.

The constraints of the system state, the control input and the control increment are as follows:

$$x_k \in \mathcal{X}, \quad k \geq 0, \tag{2a}$$
$$u_k \in \mathcal{U}, \quad k \geq 0, \tag{2b}$$
$$\Delta u_k \in \Delta\mathcal{U}, \quad k \geq 0, \tag{2c}$$

where $\mathcal{X}$ is the admissible set of the system state, $\mathcal{U}$ and $\Delta\mathcal{U}$ are the admissible sets of control input and control increment.

In this paper, we assume that all states x_k are measured instantaneously and there is neither external disturbance nor model perturbation at all.

The following assumptions are required for system (1):

Assumption 1. $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \longrightarrow \mathbb{R}^{n_x}$ is twice continuously differentiable, $f(0,0) = 0$. That is, $(0,0) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_u}$ is an equilibrium of system (1).

Assumption 2. $\mathcal{X} \subset \mathbb{R}^{n_x}$, $\mathcal{U} \subset \mathbb{R}^{n_u}$ and $\Delta\mathcal{U} \subset \mathbb{R}^{n_u}$ are compact, and the point $(0,0)$ lies in the interior of the set $\mathcal{X} \times \mathcal{U}$.

Assumption 3. There exists a n_u-dimension ball $\mathcal{P}_r \subseteq \Delta\mathcal{U}$ such that $\|\vartheta\| \leq r$ for all $\vartheta \in \mathcal{P}_r$, where $r > 0$ is a constant.

Note that Assumption 3 ensures that the set $\mathcal{P}_r$ is covered entirely by the control incremental set $\Delta U \in \mathbb{R}^{n_u}$.

At the time instant k, define the sequence of the control input

$$U_k := \{u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k}\}. \tag{3}$$

The open-loop optimization problem of the finite horizon model predictive control at the time instant k is formulated as follows:

Problem 4.

$$\underset{U_k}{\text{minimize}}\ J(x_k, U_k)$$

subject to:

$$x_{k+i+1|k} = f(x_{k+i|k}, u_{k+i|k}), \quad x_{k|k} = x_k, \tag{4a}$$
$$x_{k+i|k} \in \mathcal{X}, \quad i \in \mathbb{Z}_{[1,N-1]}, \tag{4b}$$
$$u_{k+i|k} \in \mathcal{U}, \quad i \in \mathbb{Z}_{[0,N-1]}, \tag{4c}$$
$$\Delta u_{k+i|k} \in \Delta\mathcal{U}, \quad i \in \mathbb{Z}_{[1,N-1]}, \tag{4d}$$
$$Kx_{k+N|k} - u_{k+N-1|k} \in \Delta\mathcal{U}, \tag{4e}$$
$$x_{k+N|k} \in \Omega, \tag{4f}$$

where

$$J(x_k, U_k) = \sum_{i=0}^{N-1} \|x_{k+i|k}\|_Q^2 + \|u_{k+i|k}\|_R^2 + \|x_{k+N|k}\|_P^2 \tag{5}$$

and Q and R are positive definite matrices. The terminal set Ω and the terminal penalty $\|x_{k+N|k}\|_P^2$ will be introduced in the future.

Remark 5. In order to guarantee recursive feasibility, an extra terminal constraint $Kx_{k+N|k} - u_{k+N-1|k} \in \Delta\mathcal{U}$ is imposed on Problem 4.

When $i > 0$, the control increment $\Delta u_{k+i+1|k} := u_{k+i+1|k} - u_{k+i|k}$. Furthermore,

$$\Delta u_{k|k} = \begin{cases} u_{k|k} - u_{k-1}, & \text{if } k \geq 1, \\ u_{k|k}, & \text{if } k = 0. \end{cases}$$

The control objective of finite horizon MPC is to achieve a finite horizon control sequence by solving Problem 4 online such that system (1) is stable, the performance (5) is minimized and the constraints (2) are satisfied.

Although a finite horizon control sequence is achieved by solving Problem 4, only the first control action in the sequence is applied to the considered systems. At the next time instant, the whole process will be repeated with the new measurement of the system states.

At the time instant k, the control sequence U_k is a feasible solution to Problem 4 if

(i) for all $i \in \mathbb{Z}_{[0,N-1]}$, the constraints (4b)–(4d) are satisfied;
(ii) the terminal constraints $x_{k+N|k} \in \Omega$ and $Kx_{k+N|k} - u_{k+N-1|k} \in \Delta\mathcal{U}$ are satisfied;
(iii) the cost function (5) is finite, i.e.,

$$\sum_{i=0}^{N-1} \|x_{k+i|k}\|_Q^2 + \|u_{k+i|k}\|_R^2 + \|x_{k+N|k}\|_P^2 < \infty.$$

Suppose that Problem 4 has an optimal solution U_k^* at the time instant k,

$$U_k^* := \{u_{k|k}^*, u_{k+1|k}^*, \ldots, u_{k+N-1|k}^*\}, \tag{6}$$

and the corresponding optimal trajectory is denoted as

$$X_k^* := \{x_{k+1|k}^*, x_{k+2|k}^*, \ldots, x_{k+N|k}^*\}. \tag{7}$$

As only the first control action in the sequence U_k^* is applied to the system, the actual control at the time instant k is

$$u_k := u_{k|k}^*.$$

The Jacobian linearization of the nonlinear system (1) at the equilibrium $(0, 0)$ is

$$\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k \tag{8}$$

with $A := \frac{\partial f}{\partial x}|_{(0,0)}$ and $B := \frac{\partial f}{\partial u}|_{(0,0)}$.

Assumption 6. System (8) is stabilizable.

That is, there exists a linear state feedback control law $u = Kx$ such that the controlled system $A_k := A + BK$ is asymptotic stability. Without loss of generality, assume that A_k has at least a nonzero eigenvalue. For such a given K, the following lemma can be proven.

Lemma 7. *Suppose that Assumptions 1–3 and 6 are satisfied. Then,*
(1) Lyapunov equation

$$\kappa^2 A_k^T P A_k - P = -(Q + K^T R K) \tag{9}$$

admits a unique positive definite solution P, where $\kappa \in \left(1, \frac{1}{\tau_{\max}(A_k)}\right)$ and $\tau_{\max}(A_k)$ is the maximum magnitude of the eigenvalues of matrix A_k.
(2) There exists $\alpha > 0$ which specifies an ellipsoid

$$\Omega := \left\{x \in \mathbb{R}^{n_x} \mid x^T P x \le \alpha\right\},$$

such that

(i) $\Omega \subseteq \mathcal{X}$,
(ii) $Kx \in \mathcal{U}$ *for all* $x \in \Omega$,
(iii) $K(x_1 - x_2) \in \Delta\mathcal{U}$ *for all* $x_1, x_2 \in \Omega$,
(iv) Ω *is positive invariant for the nonlinear system (1) with the linear control* $u = Kx$,
(v) *for any* $x \in \Omega$, *and for the nonlinear system (1) with the linear control law* $u = Kx$

$$\sum_{i=0}^{\infty} \|x_{i|0}\|_Q^2 + \|u_{i|0}\|_R^2 \le x^T P x, \tag{10}$$

where $x_{0|0} = x$.

Proof. (1) For all $\kappa \in \left(1, \frac{1}{\tau_{\max}(A_k)}\right)$, all the eigenvalues of κA_k lie in the unit circle since A_k is discrete-time stable. Thus, Lyapunov function has a unique positive definite solution as $Q + K^T RK$ is positive definite.

(2) (i)–(iii) Since the equilibrium $(0, 0) \in \mathcal{X} \times \mathcal{U}$, there is a positive constant α_0 such that in the set

$$\Omega_0 := \{x \in \mathbb{R}^{n_x} \mid x^T P x \le \alpha_0\},$$

the state constraints (2a) and control constraints (2b) are satisfied.

Next, we will show that there exists a constant $\alpha_1 \in (0, \alpha_0]$ such that the incremental input constraints (2c) are satisfied in the set

$$\Omega_1 := \{x \in \mathbb{R}^{n_x} \mid x^T P x \le \alpha_1\}.$$

Denote x_1 and x_2 as two arbitrarily chosen points in the set Ω_1, then $\|x_1 - x_2\| \le 2\sqrt{\frac{\alpha_1}{\lambda_{\min}(P)}}$.

For all $k \ge 0$,

$$\begin{aligned}\|\Delta u_k\| &= \|K(x_k - x_{k-1})\| \\ &\le \|K\| \cdot \|x_k - x_{k-1}\| \\ &\le \|K\| \cdot \max_{\forall x_1, x_2 \in \Omega_1} \|x_1 - x_2\| \\ &\le \|K\| 2\sqrt{\frac{\alpha_1}{\lambda_{\min}(P)}}.\end{aligned}$$

Denote

$$\alpha_1 = \min\left\{\alpha_0, \frac{r^2 \lambda_{\min}(P)}{4\|K\|^2}\right\},$$

then $\|\Delta u_k\| \le r$ for all $k \ge 0$.

Furthermore, the incremental constraints (2c) are satisfied, since $\Delta u_k \in \mathcal{P}_r$ if $\|\Delta u_k\| \le r$, and $\Delta u_k \in \Delta\mathcal{U}$ if $\Delta u_k \in \mathcal{P}_r$ in accordance with Assumption 3.

(iv) The time difference of $x^T P x$ along the trajectory $x_{k+1} = f(x_k, Kx_k)$ is

$$\begin{aligned} &x_{k+1}^T P x_{k+1} - x_k^T P x_k \\ &\quad = (A_k x_k + \phi_k)^T P (A_k x_k + \phi_k) - x_k^T P x_k \\ &\quad = \kappa^2 x_k^T A_k^T P A_k x_k - x_k^T P x_k + (1 - \kappa^2) x_k^T A_k^T P A_k x_k \\ &\qquad + x_k^T A_k^T P \phi_k + \phi_k^T P A_k x_k + \phi_k^T P \phi_k, \end{aligned} \tag{11}$$

with $\phi_k := f(x_k, Kx_k) - A_k x_k$.

Furthermore, there exists $\xi_k := \varsigma \cdot 0 + (1 - \varsigma) x_k$ for some $\varsigma \in (0, 1)$ such that

$$\begin{aligned}\phi_k &= \frac{1}{2} x_k^T \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial x_k^2} x_k + \frac{1}{2} x_k^T \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial x_k \partial u_k} K x_k \\ &\quad + \frac{1}{2} x_k^T K^T \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial u_k \partial x_k} x_k \\ &\quad + \frac{1}{2} x_k^T K^T \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial u_k^2} K x_k\end{aligned}$$

from the Mean Value Theorem.

Denote

$$\begin{aligned}C_M(\xi_k) &:= \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial x_k^2} + \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial x_k \partial u_k} K \\ &\qquad + K^T \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial u_k \partial x_k} + K^T \frac{\partial^2 f(\xi_k, K\xi_k)}{\partial u_k^2} K.\end{aligned}$$

Since f is twice continuously differentiable, $C_M(\cdot)$ is continuous in the set Ω_1. For simplicity, denote

$$C_{\max} := \sup_{\xi_k \in \Omega_1} \|C_M(\xi_k)\|.$$

Thus, $\|\phi_k\| \le \frac{1}{2}C_{\max}\|x_k\|^2$ for all $x_k \in \Omega_1$. Furthermore,

$$\phi_k^T P\phi_k + x_k^T A_k^T P\phi_k + \phi_k^T PA_k x_k \le \frac{1}{4}C_{\max}^2\|P\|\|x_k\|^4 + C_{\max}\|A_k\|\|P\|\|x_k\|^3. \quad (12)$$

Choose an $\alpha \in (0, \alpha_1]$ such that

$$\frac{1}{4}C_{\max}^2\|P\|\|x_k\|^2 + C_{\max}\|A_k\|\|P\|\|x_k\| \le (\kappa^2 - 1)\lambda_{\min}(A_k^T PA_k)$$

for all $x_k \in \Omega$, see Lemma 16 in the Appendix.

Then, for all $x_k \in \Omega$,

$$\phi_k^T P\phi_k + x_k^T A_k^T P\phi_k + \phi_k^T PA_k x_k \le (\kappa^2 - 1)\lambda_{\min}(A_k^T PA_k)\|x_k\|^2. \quad (13)$$

Since $x_k^T A_k^T PA_k x_k \ge \lambda_{\min}(A_k^T PA_k)\|x_k\|^2$, one has

$$\phi_k^T P\phi_k + x_k^T A_k^T P\phi_k + \phi_k^T PA_k x_k \le (\kappa^2 - 1)x_k^T A_k^T PA_k x_k. \quad (14)$$

Substituting (14) into (11) yields that

$$x_{k+1}^T Px_{k+1} - x_k^T Px_k \le \kappa^2 x_k^T A_k^2 PA_k x_k - x_k^T Px_k. \quad (15)$$

Using the Lyapunov equation (9), one has then

$$x_{k+1}^T Px_{k+1} - x_k^T Px_k \le -x_k^T(Q + K^T RK)x_k. \quad (16)$$

Since $P > 0$ and $Q + K^T RK > 0$, nonlinear system (1) with the linear control law $u = Kx$ is asymptotically stable. Furthermore, inequality (16) implies that the set Ω is invariant for nonlinear system (1) with the linear control law $u = Kx$.

(v) For any $x \in \Omega$, adding up (16) from 0 to ∞ with initial condition $x_{i|0} = x$ yields the desired results (10).

Note that the satisfaction of Eq. (13) for all $x_k \in \Omega$ guarantees that Ω is an invariant set for the original nonlinear systems with the linear control law $u = Kx$.

3. Properties of the optimal cost function

Suppose that the optimization problem has a feasible solution at $x \in \mathcal{X}$, and define the corresponding cost function as

$$E(x) := \min_{U_k} J(x, U_k).$$

The optimal cost function has the following properties.

Theorem 8. *Considering the discrete-time nonlinear systems, the optimal cost function $E(x)$ has the properties as follows:*

(i) *$E(0) = 0$ and $E(x) > 0$ for all $x \neq 0$,*
(ii) *$E(x)$ is continuous at $x = 0$,*
(iii) *$E(x)$ is monotonically decreasing along the predicted trajectory, and*

$$E(x_{k+1}) \le E(x_k) - \|x_k\|_Q^2 - \|u_k\|_R^2. \quad (17)$$

Proof. (i) Since $Q > 0$ and $R > 0$, $E(x) > 0$ as $x \neq 0$. Given $x_k = 0$, the feasible solution is $u_{k+i|k}^* \equiv 0$, and the corresponding optimal predicted trajectory is $x_{k+i+1|k}^* \equiv 0$, for all $i \in [0, N-1]$. Thus, $E(x_k) = 0$ as $x_k = 0$. In accordance with $R > 0$,

$$E(x_k) \ge \sum_{i=0}^{N-1}\|x_{k+i|k}^*\|_Q^2 + \|x_{k+N|k}^*\|_P^2.$$

Obviously, $E(x_k) = 0$ can be achieved only if $x_{k+i|k}^* \equiv 0$ for all $i \in [0, N]$. Therefore, for all $x_k \neq 0$, $E(x_k) > 0$.

(ii) In order to show the continuity of $E(x)$ at the equilibrium $x = 0$, pick up a point $x_k \in \Omega$ with $x_k \neq 0$. Then,

$$\bar{U}_k = \left\{Kx_{k|k}, Kx_{k+1|k}, \ldots, Kx_{k+N-1|k}\right\} \quad (18)$$

is a feasible solution to the optimization problem. In accordance with Lemma 7, $\bar{E}(x_k) := x_k^T Px_k$ is an upper bound of the finite horizon cost function. That is, $E(x_k) \le \bar{E}(x_k)$ for all $x_k \in \Omega$.

Since $\bar{E}(x)$ is twice continuously differentiable, $\bar{E}(x)$ is continuous at $x = 0$. For any $\epsilon > 0$, there exists $\delta_0 > 0$ such that $|\bar{E}(x) - \bar{E}(0)| \le \epsilon$ as $\|x - 0\| \le \delta_0$.

Define a set

$$B_0 := \left\{x \in \mathbb{R}^{n_x} \mid x^T x \le \frac{\alpha}{\lambda_{\max}(P)}\right\},$$

$B_0 \subseteq \Omega$. Denote $\delta := \min\left\{\delta_0, \sqrt{\frac{\alpha}{\lambda_{\max}(P)}}\right\}$, then $|E(x) - E(0)| \le \epsilon$ for all $\|x - 0\| \le \delta$. Therefore, the optimal cost function $E(x)$ is continuous at $x = 0$.

(iii) Suppose that at the time instant k, the optimization problem has a feasible solution U_k^*, which satisfies the control constraints (4c) and the control incremental constraints (4d). The corresponding predicted state sequence X_k^* satisfies state constraints (4b), and $x_{k+N|k}^* \in \Omega$. Furthermore, the extra terminal constraint $Kx_{k+N|k}^* - u_{k+N-1|k}^* \in \Delta\mathcal{U}$ is satisfied. The control sequence (6) guarantees that the cost function

$$E(x_k) = \sum_{i=0}^{N-1}\|x_{k+i|k}^*\|_Q^2 + \|u_{k+i|k}^*\|_R^2 + \|x_{k+N|k}^*\|_P^2$$

is finite.

Implement the control $u_k = u_{k|k}^*$ into the systems (1). The system state at the time instant $k+1$ is

$$x_{k+1} = f(x_k, u_{k|k}^*).$$

Since neither model perturbations nor external exogenous is considered, $x_{k+1} = x_{k+1|k}^*$.

At the time instant $k+1$, choose

$$U_{k+1} \triangleq \left[u_{k+1|k}^*, \quad \ldots, \quad u_{k+N-1|k}^*, \quad Kx_{k+N|k}^*,\right], \quad (19)$$

as a feasible solution to the optimization problem, which is a shifted control sequence of the one obtained at the time instant k followed by the linear control law. Since $x_{k+N|k}^* \in \Omega$, $Kx_{k+N|k}^* \in \mathcal{U}$. Thus, the input constraint is satisfied for U_{k+1}.

In accordance with U_{k+1}, the state sequence is

$$\begin{cases} x_{k+1+i|k+1} = x_{k+1+i|k}^*, & i \in [1, N-1], \\ x_{k+1+N|k+1} = f(x_{k+N|k}^*, Kx_{k+N|k}^*), & i = N, \end{cases}$$

which satisfies the state constraints. As Ω is invariant under the linear control law, the terminal constraint (4f) is satisfied. Since $Kx_{k+N|k}^* - u_{k+N-1|k}^* \in \Delta\mathcal{U}$ and the linear control law Kx satisfies the incremental input constraints for all $x \in \Omega$, the incremental constraints (4d) and the extra terminal constraint (4e) are satisfied at the time instant $k+1$.

The cost function at the time instant $k+1$ is

$$\begin{aligned} J_{k+1} &= \sum_{i=0}^{N-1}\left\|x_{k+1+i|k+1}\right\|_Q^2 + \left\|u_{k+1+i|k+1}\right\|_R^2 + \left\|x_{k+1+N|k+1}\right\|_P^2 \\ &= \sum_{i=0}^{N-2}\left\|x_{k+1+i|k}^*\right\|_Q^2 + \left\|u_{k+1+i|k}^*\right\|_R^2 \\ &\quad + \left\|x_{k+N|k+1}\right\|_Q^2 + \left\|u_{k+N|k+1}\right\|_R^2 + \|x_{k+1+N|k+1}\|_P^2 \\ &= \sum_{i=0}^{N-1}\left\|x_{k+i|k}^*\right\|_Q^2 + \left\|u_{k+i|k}^*\right\|_R^2 - \left\|x_{k|k}^*\right\|_Q^2 \end{aligned}$$

$$
\begin{aligned}
&- \left\|u^*_{k|k}\right\|^2_R + \left\|x^*_{k+N|k}\right\|^2_Q + \left\|Kx^*_{k+N|k}\right\|^2_R \\
&+ \left\|f(x^*_{k+N|k}, Kx^*_{k+N|k})\right\|^2_P \\
= {}& E(x_k) - \|x_k\|^2_Q - \|u_k\|^2_R - \left\|x^*_{k+N|k}\right\|^2_P \\
&+ \left\|f(x^*_{k+N|k}, Kx^*_{k+N|k})\right\|^2_P + \left\|x^*_{k+N|k}\right\|^2_{Q+K^TRK}.
\end{aligned}
$$

As $x^*_{k+N|k} \in \Omega$, Eq. (16) is satisfied for nonlinear system (1) with the linear control law $u = Kx$. Thus,

$$J_{k+1} \le E(x_k) - \|x_k\|^2_Q - \|u_k\|^2_R.$$

Since $E(x_k)$ is finite, J_{k+1} is finite. Thus, the candidate control sequence (19) is a feasible solution to Problem 4 at the time instant $k+1$. Furthermore, since the optimal solution is better than the feasible solution,

$$E(x_{k+1}) \le J_{k+1} \le E(x_k) - \|x_k\|^2_Q - \|u_k\|^2_R. \tag{20}$$

Therefore, $E(x)$ is monotonically decreasing along the system trajectory.

Remark 9. In general, neither the optimal control law nor the optimal cost function is necessarily continuous in the system states, which is shown by Meadows, Henson, Eaton, and Rawlings (1995) with an example. Thus, asymptotic stability cannot be obtained directly, since the candidate Lyapunov function is expected to be continuous in the classical Lyapunov Theorem.

Remark 10. Kellet and Teel revealed that the nonlinear system $x_{k+1} = f(x_k, \kappa(x_k))$ is inherently robustly stable if and only if it admits a continuous Lyapunov function (Kellet & Teel, 2004). The considered system under model predictive control law may be unstable for any small disturbance or perturbation, since the optimal cost function chosen as the candidate Lyapunov function may be discontinuous in the system states (Grimm, Messina, Tuna, & Teel, 2004).

From the deduction of Theorem 8, it is easy to see that if the optimization problem has a feasible solution at the time instant k, then the optimization problem has a feasible solution at the time instant $k+1$. Thus, we come to the following conclusion:

Corollary 11. *Suppose that Problem 4 has a feasible solution at the time instant* $k = 0$*, Problem 4 has a feasible solution at each time instant* $k > 0$.

Proof. Due to the space limitation, it is omitted.

4. Properties of systems under control

In this section, properties of systems under model predictive control are discussed. Convergence to the equilibrium is proven by the monotonically decreasing of the optimal cost function, and Lyapunov stability is proven by a candidate Lyapunov function which is locally continuous. Asymptotic stability is proven in accordance with convergence together with Lyapunov stability locally. Furthermore, compactness of feasible sets is discussed.

4.1. Asymptotic stability of systems

Lemma 12. *Suppose that*

(i) *Assumptions 1–3* and 6 *are satisfied,*
(ii) *At the time instant* $k = 0$*, Problem 4 has a feasible solution,*

then,

(1) x_k *converges to the set* Ω *in finite time,*
(2) $\lim_{k\to\infty} x_k = 0$.

Proof. (1) Firstly, it is proven by reduction to absurdity that there exists a finite positive constant N such that $x_k \in B_0$ when $k > N$. Let $\Phi(i; x)$ be the solution of the system under control that starts from initial state x at time i, and $\Phi(0; x) = x$. Suppose contrary to what is to be proven that $\Phi(k; x) \notin B_0$ for all $k > 0$. Eq. (17) implies that

$$E(\Phi(k+1; x)) \le E(\Phi(k; x)) - \|\Phi(k; x)\|^2_Q. \tag{21}$$

By iterating Eq. (21) from 1 to $k-1$, it is obtained that

$$E(\Phi(k; x)) \le -\sum_{i=1}^{k-1} \|\Phi(i; x)\|^2_Q + E(x).$$

Since $\|\Phi(i; x)\| > \sqrt{\frac{\alpha}{\lambda_{\max}(P)}}$ for all $i > 0$,

$$E(\Phi(k; x)) < -\sum_{i=1}^{k-1} \frac{\alpha\lambda_{\min}(Q)}{\lambda_{\max}(P)} + E(x).$$

Thus, $E(\Phi(k; x)) \xrightarrow{k\to\infty} -\infty$ since $\frac{\alpha\lambda_{\min}(Q)}{\lambda_{\max}(P)} > 0$ is constant, which contradicts with $E(x) \ge 0$ obviously.

(2) According to inequality (21), $E(x)$ is monotonically non-increasing. Furthermore, the lower bound of $E(x)$ is $E(0) = 0$. Thus, $E(x)$ is convergent as $k \to \infty$ for bounded and monotonic sequence has a finite limit. By taking limits on both sides of inequality (21), we have

$$\lim_{k\to\infty} \|x_k\|^2_Q \le \lim_{k\to\infty} E(x_k) - \lim_{k\to\infty} E(x_{k+1}) = 0.$$

Thus, $x_k \to 0$ as $t \to \infty$.

The optimal cost function was firstly employed in Keerthi and Gilbert (1988) as a Lyapunov function for establishing stability of model predictive control of constrained time-varying nonlinear discrete-times systems. Thereafter, the optimal cost function was almost universally employed as a natural Lyapunov function for stability analysis of model predictive control (Mayne et al., 2000).

The main difficulty to choose the optimal cost function as a candidate Lyapunov function in the analysis of the stability is that it is hard to show the continuity of it on the state x. In principle, the concept of stability only reflects a local property of the system at the equilibrium. Thus, in the following

$$V(x) := x^T Px, \quad \forall x \in \Omega,$$

is chosen as the candidate Lyapunov function which is an upper bound of the optimal cost function $E(x)$, and continuously differentiable in x.

Theorem 13. *Suppose that*

(i) *Assumptions 1–3* and 6 *are satisfied,*
(ii) *At the time instant* $k = 0$*, Problem 4 has a feasible solution,*

then, the closed-loop system is asymptotically stable.

Proof. (i) Since $V(0) = 0$, $V(x) > 0$ for all $x \in \Omega$ with $x \ne 0$, and

$$V(x_{k+1}) - V(x_k) \le -x_k^T \left(Q + K^T RK\right) x_k,$$

the system under finite horizon model predictive control is stable in accordance with Haddad and Chellaboina (2007, Theorem 13.2).

(ii) Since x_k converges to Ω in finite time if Problem 4 has a feasible solution at the time instant $k = 0$ and $\lim_{k\to\infty} x_k = 0$, the system is asymptotically stable.

Remark 14. In finite horizon MPC, a N-step control sequence is obtained by solving the optimization problem at each time instant which drives the system state to the terminal set and the terminal constraint is satisfied. With the N-step control sequence (a feasible solution to the optimization problem), monotonic decrease of the cost function can be guaranteed. Thus, stability of the controlled systems cannot be affected or altered if only a local optimum is obtained.

4.2. Compactness of feasible sets

Denote $\mathcal{X}_j$ as the set of states in $\mathcal{X}$ that can be steered to the terminal set Ω in j steps by an admissible control sequence

$$\mathcal{X}_j := \left\{ x \in \mathcal{X} \mid \exists u \in \mathcal{U} \text{ such that } f(x, u) \in \mathcal{X}_{j-1} \right\}$$

for $j = 1, 2, \ldots, N$ (j is time to go), with the terminal condition $\mathcal{X}_0 = \Omega$.

As $\mathcal{X}_j \subseteq \mathcal{X}$ in which $\mathcal{X}$ is bounded, $\mathcal{X}_j$ is bounded. In accordance with Rawlings and Mayne (2009, Proposition 2.11), $\mathcal{X}_j$ is closed. Thus, $\mathcal{X}_j$ is a compact set for $j = 1, 2, \ldots, N$.

Remark 15. To estimate or determine the domain of attraction of the involved MPC problem is not a trivial work, cf. Genesio, Tartaglia, and Vicino (1985) and Chesi (2009). In principle, the domain of attraction relies on the nonlinear properties of considered systems, in particular, controllability of it.

5. Numerical example

In this section, a numerical example is investigated in order to verify the effectiveness of the proposed method. The system is described by

$$\begin{aligned} x_{k+1}^{(1)} &= x_k^{(1)} + 0.1x_k^{(2)} + 0.1u_k \left(\mu + (1-\mu)x_k^{(1)} \right), \\ x_{k+1}^{(2)} &= x_k^{(2)} + 0.1x_k^{(1)} + 0.1u_k \left(\mu - 4(1-\mu)x_k^{(2)} \right), \end{aligned} \tag{22}$$

with $\mu = 0.5$ and $x_k = \begin{bmatrix} x_k^{(1)} & x_k^{(2)} \end{bmatrix}^T$. The linearized system of (22) is unstable and controllable.

Assume that x_k can be measured, and then the input constraint and the incremental input constraint are

$$\begin{aligned} -2 \le u_k \le 2, \quad k \ge 0, \\ -0.5 \le \Delta u_k \le 0.5, \quad k \ge 0. \end{aligned}$$

The weighting matrices are chosen as $Q = \text{diag}\{0.5, 0.5\}$, $R = 1$. First, a linear state feedback control matrix $K = [-2.0107 - 2.0107]$ is got by solving a linear quadratic regulator (LQR) problem with the weighting matrices for the locally linearized systems. Note that other linear control laws which stabilize the locally linearized systems can also be used to get a linear state feedback control matrix. The largest eigenvalue in the sense of magnitude of the closed-loop linear systems is $\lambda(A_k) = 0.9$. Then, a constant $\kappa = 1/0.91$ is chosen to solve the Lyapunov equation (9). The unique solution of (9) is $P = \begin{bmatrix} 188.9674 & 166.0917 \\ 166.0917 & 188.9674 \end{bmatrix}$, which is positive definite and can be used as a terminal penalty matrix. A conservative terminal set $\Omega = \left\{ x \in \mathbb{R}^2 \mid x^T P x \le 0.778 \right\}$ is derived from trail and error which satisfies input constraints, incremental constraints, state constraints and Eq. (13).

Fig. 1 shows the state trajectory, input and the variation of the input of the considered system starting from $x_0 = [-3.0\ 2.0]'$. The solid line shows the trajectory of the systems with the proposed scheme, and the dash-dotted line shows the trajectory of the systems with the traditional MPC scheme (Mayne et al., 2000). The prediction horizon is $N = 20$ for both the proposed scheme and the traditional MPC scheme without the consideration of incremental constraints. From Fig. 1, it can be seen that the incremental constraint is violated at the first time instant if it is not considered in the optimization problem.

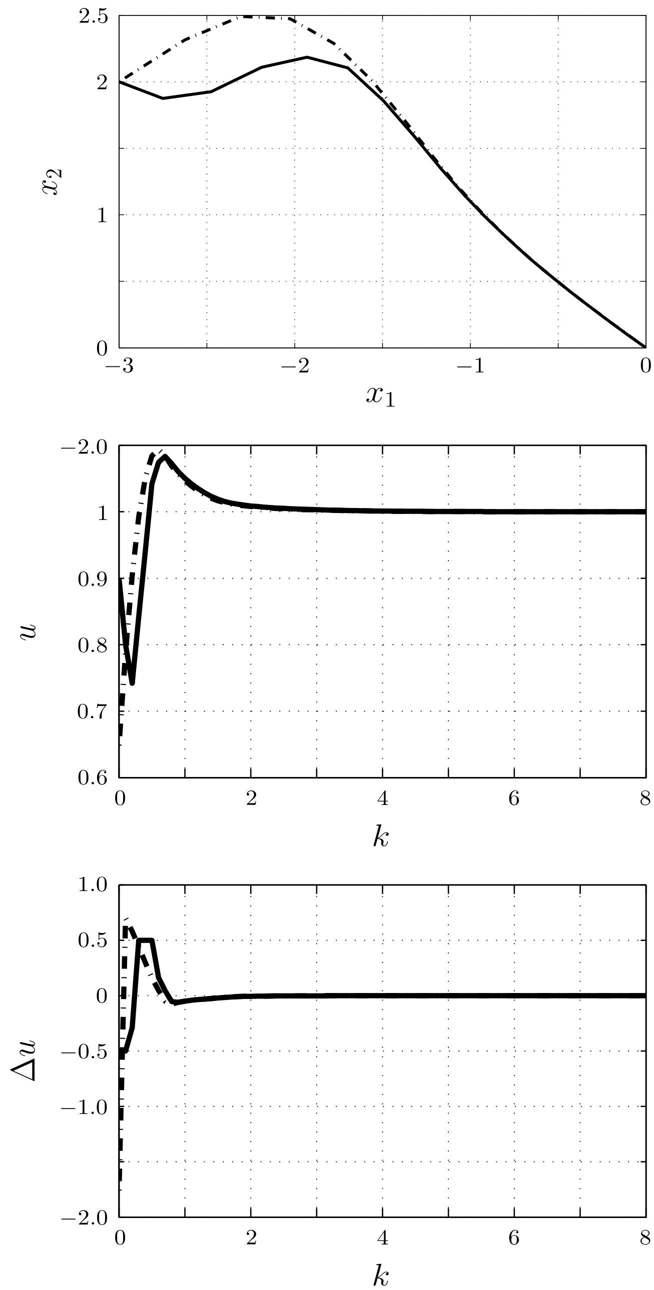

Fig. 1. Comparison of the system dynamics and control input with and without incremental input constraint, dash-dotted line: without incremental input constraint, solid line: with incremental input constraint.

6. Conclusion

Incremental control constraint reflects the allowed change rate of control input. In this paper, asymptotic stability of finite horizon MPC with incremental input constraints was studied. The existence of the terminal set, terminal penalty and terminal control law was proven under the condition that the considered systems are twice continuously differentiable. The optimal cost function is positive semi-definite, continuous at the equilibrium and monotonically decreasing along the system trajectory. The system state will converge to the equilibrium since the optimal cost function is monotonically decreasing. Stability was proven by choosing the upper bound of the optimal cost function in the terminal set as a candidate control Lyapunov function. Furthermore, the system is asymptotically stable since the system state converges to the equilibrium and the system is stable.

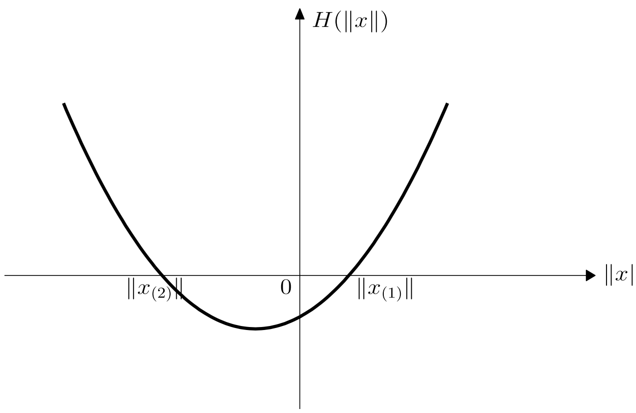

Fig. 2. $H(\|x\|)$ of $\|x\|$.

Appendix

Lemma 16. *There exists* $\alpha \in (0, \alpha_1]$ *such that* $H(\|x\|) \le 0$ *for all* $x \in \Omega$*, where*

$$H(\|x\|) := \frac{C_{\max}^2 \|P\|}{4}\|x\|^2 + C_{\max}\|A_k\|\|P\|\|x\| - \left(\kappa^2 - 1\right)\lambda_{\min}(A_k^T P A_k).$$

Proof. The equation $H(\|x\|) = 0$ has two solutions

$$\|x_{(1)}\| := -\frac{2\|A_k\|\|P\|}{C_{\max}\|P\|} + \frac{2\sqrt{\|A_k\|^2\|P\|^2 + \|P\|(\kappa^2-1)\lambda_{\min}(A_k^T P A_k)}}{C_{\max}\|P\|},$$

and

$$\|x_{(2)}\| := -\frac{2\|A_k\|\|P\|}{C_{\max}\|P\|} - \frac{2\sqrt{\|A_k\|^2\|P\|^2 + \|P\|(\kappa^2-1)\lambda_{\min}(A_k^T P A_k)}}{C_{\max}\|P\|}.$$

where $\|x_{(1)}\| > 0$ and $\|x_{(2)}\| < 0$. The curve of $H(\|x\|)$ of $\|x\|$ is shown in Fig. 2. Since $\|x\| \ge 0$, $H(\|x\|) \le 0$ as $\|x\| \in \left[0, \|x_{(1)}\|\right]$.

Choose $\alpha = \min\left\{\alpha_1, \|x_{(1)}\|^2\lambda_{\min}(P)\right\}$. Then, for all $x \in \Omega$, $\|x\| \le \sqrt{\frac{\alpha}{\lambda_{\min}(P)}} \le \|x_{(1)}\|$. That is, $H(\|x\|) \le 0$ for all $x \in \Omega$.

References

Angeli, D., Casavola, A., & Mosca, E. (2000). Predictive PI-Control of linear plant under positional and incermental input saturations. *Automatica*, *36*(10), 1505–1516.

Benzaouia, Abdellah, Tadeo, Fernando, & Mesquine, Fouad (2006). The regulator problem for linear systems with saturations on the control and its increments or rate: An LMI approach. *IEEE Transactions on Circuits and Systems*, *53*(12), 2681–2691.

Berg, J. M., Hammet, K. D., Schwartz, C. A., & Banda, S. S. (1996). An analysis of the destabilizing effect of daisy chained rate-limited actuators. *IEEE Transactions on Control Systems Technology*, *4*(2).

Chen, H. (1997). *Stability and robustness considerations in nonlinear model predictive control*. (Ph.D. Thesis), Germany: University of Stuttgart.

Chen, H., & Allgöwer, F. (1998). A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica*, *34*(10), 1205–1217.

Chesi, Graziano (2009). Estimating the domain of attraction for non-polynomial sysems via LMI optimizations. *Automatica*, *45*(6), 1536–1541.

Ding, B. C. (2009). Model predictive control for nonlinear systems represented by a Takagi–Sugeno model. *Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications & Algorithms*, *16*(6B), 859–886.

Genesio, Roberto, Tartaglia, Michele, & Vicino, Antonio (1985). On the estimation of asymptotic stability regions: State of the art and new properties. *IEEE Transactions on Automatic Control*, *30*(8), 747–755.

Gonzalez, Alegandro H., & Odloak, Darci (2011). *Robust model predictive control for time delayed systems with optimizing targets and zone control*. (pp. 339–373). INTECH Open Access Publisher.

Gonzalez, Alegandro H., Perez, Jose M., & Odloak, Darci (2009). Infinite horizon MPC with non-minimal state space feedback. *Journal of Process Control*, *19*(3), 473–481.

Grimm, G., Messina, M. J., Tuna, S. E., & Teel, A. R. (2004). Examples when nonlinear model predictive control is nonrobust. *Automatica*, *40*(10), 1729–1738.

Haddad, W. M., & Chellaboina, V. (2007). *Nonlinear dynamical systems and control: A Lypunov-based approach*. Princeton, New Jersey: Princeton University Press.

Keerthi, S. S., & Gilbert, E. G. (1988). Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving-horizon approximations. *Journal of Optimization Theory and Applications*, *57*(2), 265–293.

Kellet, C. M., & Teel, A. R. (2004). Smooth Lyapunov function and robustness of stability for differential inclusions. *Systems & Control Letters*, *52*, 395–405.

Lin, Z. (1997). Semi-global stabilization of linear systems with position and rate-limited actuators. *Systems & Control Letters*, *30*, 1–11.

Mayne, D. Q., Rawlings, J. B., Rao, C. V., & Scokaert, P. O. M. (2000). Constrained model predictive control: stability and optimality. *Automatica*, *36*(6), 789–814.

Meadows, E. S., Henson, M. A., Eaton, J. W., & Rawlings, J. B. (1995). Receding horizon control and discontinuous state feedback stabilization. *International Journal of Control*, *62*(5), 1217–1229.

Mesquine, F., Tadeo, F., & Benzaouia, A. (2004). Regulator problems for linear systems with constraints on control and its increment or rate. *Automatica*, *40*(8), 1387–1495.

Mesquine, F., Tadeo, F., & Benzaouia, A. (2006). Constrained control and rate or increment for linear systems with additive disturbances. *Mathematical Problems in Engineering*, 1–16.

Mosca, E., & Tesi, Pietro (2008). Horizon-switching prictive set-point tracking under input-increment saturations and persistent disturbances. *European Journal of Control*, *14*(1), 5–15.

Mosca, E., Tesi, Pietro, & Zhang, J. (2008). Feasbility of horizon-switching predictive control under positional and incremental input saturation. *Automatica*, *44*(11), 2936–2939.

Rawlings, J. B., & Mayne, D. Q. (2009). *Model predictive control: Theory and design*. Madison, Wisconsin: Nob Hill Publishing.

Rawlings, J. B., & Muske, K. R. (1993). Stability of constrained receding horizon control. *IEEE Transactions on Automatic Control*, *AC-38*(10), 1512–1516.

Rodrigues, Marco A., & Odloak, Darci (2003). An infinie horizon model predictive control for stable and integrting processes. *Computers and Chemical Engineering*, *27*(8–9), 1113–1128.

Tyan, F., & Bernstein, D. S. (1997). Dynamic output feedback compensation for linear systems with independent amplitude and rate saturations. *International Journal of Control*, *67*(1), 89–116.

Wang, Liuping (2009). *Model predictive control system design and implementation using Mtlab*. London: Springer.

Yu, Shuyou, Qu, Ting, & Chen, Hong (2013). Stability of infinite horizon MPC with incremental input constraints. In *Proceedings of the 32nd Chinese control conference* (pp. 4125–4130).

Yu, Shuyou, Qu, Ting, Xu, Fang, & Chen, Hong (2015). A revisit to MPC of discrete-time nonlinear systems. In *The annual IEEE international conference on cyber technology in automation, control and intelligent systems*, Shenyang China (pp. 7–12).

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